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On the polarization of unstable D0-branes into non-commutative odd spheres

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Abstract: We consider the polarization of unstable type-IIB D0-branes in the presence of a background five-form field strength. This phenomenon is studied from the point of view of the leading terms in the non-abelian Born Infeld action of the unstable D0-branes. The equations have SO(4) invariant solutions describing a non-commutative 3-sphere, which becomes a classical 3-sphere in the large-N limit. We discuss the interpretation of these solutions as spherical D3-branes. The tachyon plays a tantalizingly geometrical role in relating the fuzzy S^3 geometry to that of a fuzzy S^4.

Keywords: D-branes, M(atrix) Theories, Non-Commutative Geometry.

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Contents

1. Introduction

The view that zero-branes are fundamental objects that capture a number of important features of M-theory leads one to expect that higher branes can be constructed from zero-branes in type-IIA string theory [1, 2, 3]. Zero-branes of type IIA can be polarized into D2-branes in the presence of background Ramond-Ramond flux [4], leading to a natural physical context where the non-commutative 2-sphere [5] appears. The equations defining the non-commutative 2-sphere appear naturally as conditions obeyed by the matrices corresponding to transverse coordinates in the worldvolume theory of the zero-branes. In this paper, we will instead consider the polarization of unstable type-IIB D0-branes by Ramond-Ramond flux. This provides a natural physical context for obtaining odd non-commutative spheres.

Recently, much progress has been made in understanding the dynamics of unstable branes. One of the remarkable features of these systems is that stable branes can be obtained as solitons of higher-dimensional unstable branes [6, 7, 8], leading to an interpretation of brane charges in terms of K-theory. A non-abelian Born-Infeld type action for unstable D-branes has been written down [9]–[14] as have their

Chern-Simons couplings to Ramond-Ramond forms [8, 9], [16]–[19]. This action is for the most part phenomenological, satisfying certain physical requirements. Its exact form is unknown, although the tachyon potential and kinetic term have been computed exactly within the context of boundary string field theory [20], giving a result agreeing with a proposal in [13]. Using the non-abelian action of unstable D0-branes, solutions corresponding to higher-dimensional branes with a flat geometry were obtained in [21]. In the presence of a constant Ramond-Ramond five-form field strength, the D0 action gives a system of matrix equations which one might expect to have unstable solutions describing a fuzzy three-sphere.

We will find matrices ϕ^i, for $i = 1, \ldots, 4$, which define a fuzzy 3-sphere and solve the D0 equations of motion in the five-form background. These matrices satisfy $\phi^i \phi^i = R^2$ and commute in the large-N limit. Our construction is based on that which gave the fuzzy 4-sphere of [22]. The fuzzy S^3 may be interpreted as a subspace of the fuzzy S^4. The solution for the tachyon is is related to the fifth embedding coordinate involved in the definition of the fuzzy S^4. These solutions exist for matrices of size $N = (n+1)(n+3)/2$, where n is an odd number.

At finite N, the coupling of our solutions to the five-form field strength resembles the dipole moment coupling of a spherical D3-brane. However, in the large-N limit where the non-commutative S^3 becomes a classical S^3, this dipole moment vanishes for the particular class of large-N solutions we consider. Thus we do not interpret these solutions as spherical D3-branes. We expect that this situation may change if one considers a consistent supergravity background, instead of a flat geometry with five-form flux. Furthermore, if a dual D3-brane description exists, then the phenomenological lagrangian which we work with might be insufficient to obtain the correctly normalized couplings, which depend sensitively on tachyon dynamics.

2. Unstable D0-brane action

We are interested in the behavior of an unstable IIB D0-brane in a constant background RR five-form field strength

$$H^{(5)} = h dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 - h dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9 \,. \tag{2.1}$$

The low energy action for an unstable D0-brane in type-IIB string theory can be obtained by T-dualizing the unstable IIA D9-brane action as in [4]. There is a large literature on Born-Infeld and Chern-Simons terms for unstable branes [8]–[14], [16]–[19]. The D9 action is $S^9 = S^9_{\mathrm{DBI}} + S^9_{\mathrm{CS}}$ where

$$\begin{aligned} S^9_{\mathrm{DBI}} = \frac{1}{g_s \alpha'^5 (2\pi)^9} \int d^{10}x \, \mathrm{Tr} \Big[& V(T) \sqrt{\det(G + 2\pi\alpha' F)} + \\ & + \alpha' f(T) \sqrt{\det G} D_\mu T D_\mu T \Big] \,. \end{aligned} \tag{2.2}$$

Following conjectures of Sen [6], one is led to expect that $f(T)$ and $V(T)$ vanish at the global minimum of $V(T)$, for a suitable choice of the variable T. If one changes variables such that the tachyon kinetic term is canonically normalized, or $f(T) = 1$, then the Chern-Simons coupling is of the form:

$$S^9_{\mathrm{CS}} = \int \mathrm{Tr}\, DT \wedge C \wedge e^{F+B}\,. \tag{2.3}$$

A form for the potential and kinetic terms was proposed in [13] and computed exactly within the context of boundary string field theory in [20], giving

$$V(T) = \mu f(T) \tag{2.4}$$

with

$$V(T) = e^{-T^2/4} \tag{2.5}$$

and

$$\mu = \frac{1}{2ln2}\,. \tag{2.6}$$

We shall find solutions for this particular potential and kinetic term, however the existence of fuzzy three-sphere solutions does not depend on the detailed form of the potential. With a choice of variables such that the tachyon kinetic term is given by (2.4) and (2.5), the Chern-Simons coupling is[1]

$$S^9_{\mathrm{CS}} = \int \mathrm{Tr}\, \sqrt{2ln2}e^{-T^2/8} DT \wedge C \wedge e^{F+B}\,. \tag{2.7}$$

Upon T-dualizing in all 9 spatial directions, one finds the D0 action in the presence of the above five-form field strength is $S^0 = S^0_{\mathrm{DBI}} + S^0_{\mathrm{CS}}$ where

$$\begin{aligned} S_0^{\mathrm{DBI}} &= \frac{1}{g_s \alpha'^{1/2}} \int dt\, \mathrm{STr}\, V(T) \sqrt{\det(\delta_{ab} + 2\pi\alpha'[X^a, X^b])} - \\ &\quad - \alpha' f(T)[X^a, T][X^a, T] + \cdots \end{aligned} \tag{2.8}$$

and

$$S_0^{\mathrm{CS}} = \alpha'^{3/2} h \int dt\, \mathrm{STr}\, e^{-T^2/8} \Big[[X^l, T]\epsilon_{ijkl} X^i X^j X^k \Big]\,, \tag{2.9}$$

where STr indicates a symmetrized trace. The indicies i, j, k, l run from 1 to 4 and label the matrix coordinates in which the fuzzy S^3 will be embedded. We have absorbed numerical factors of order 1 in the definition of h. Since we will look for a static solution, time derivative terms have been dropped.

In terms of dimensionless adjoint scalars, $\phi^i = \sqrt{\alpha'} X^i$, and keeping only the leading terms in the DBI action for$[\phi^i, \phi^j] \ll 1$, we have:

$$\begin{aligned} S = -\frac{1}{g_s} \int dt\, \mathrm{Str}\, V(T) \left(1 - \frac{1}{4}[\phi^i, \phi^j][\phi^i, \phi^j]\right) - \big(f(T)[\phi^i, T][\phi^i, T]\big) - \\ - h e^{-T^2/8} \phi^i \phi^j \phi^k [\phi^l, T] \epsilon_{ijkl}\,. \end{aligned} \tag{2.10}$$

[1] We thank Barton Zwiebach for pointing this out.

We shall find a class of solutions for which T^2 is proportional to the identity, and $\{T, \phi^i\} = 0$. In this case, the equations of motion are

$$V'(T)\left(1 - \frac{1}{4}[\phi^i, \phi^j][\phi^i, \phi^j]\right) - f'(T)[\phi^i, T][\phi^i, T] - \big[[\phi^i, T]f(T), \phi^i\big] - \\ - \big[f(T)[\phi^i, T], \phi^i\big] - 2he^{-T^2/8}\epsilon_{ijkl}\phi^i\phi^j\phi^k\phi^l \tag{2.11}$$

and

$$\big[[\phi^i, T]f(T), T\big] + \big[f(T)[\phi^i, T], T\big] + \frac{1}{2}\big[[\phi^i, \phi^k]V(T), \phi^k\big] + \frac{1}{2}\big[V(T)[\phi^i, \phi^k], \phi^k\big] - \\ - he^{-T^2/8}\epsilon_{ijkl}\Big([\phi^j, \phi^k][\phi^l, T] + [\phi^l, T][\phi^j, \phi^k]\Big) = 0\,. \tag{2.12}$$

Note that the Chern-Simons contribution to the above equations is modified if one seeks a more general class of solutions.

3. Non-commutative S^3 solutions at $N = 4$

We will initially consider $N = 4$ and seek solutions with the ansatz

$$\phi^i = a\gamma^i + ib\gamma^i\gamma^5 \tag{3.1}$$

and

$$T = d\gamma^5\,, \tag{3.2}$$

where the γ^i are 4×4 hermitean matrices satisfying the Spin(4) Clifford algebra, and a, b and d are real numbers.

Note that there is a more general Spin(4) invariant class of solutions having

$$T = c + d\gamma^5\,. \tag{3.3}$$

For this the ansatz for T, one has

$$\begin{aligned} f(T) &= q + r\gamma^5\,, \\ f'(T) &= \omega + \lambda\gamma^5\,, \end{aligned} \tag{3.4}$$

where the quantities q, r, ω and λ depend on c and d in a manner determined by the form of the potential. More explicitly;

$$\begin{aligned} q &= \frac{1}{2}\Big(f(c+d) + f(c-d)\Big)\,, \\ r &= 1/2\Big(f(c+d) - f(c-d)\Big)\,, \\ \omega &= \frac{1}{2}\Big(f'(c+d) + f'(c-d)\Big)\,, \\ \lambda &= \frac{1}{2}\Big(f'(c+d) - f'(c-d)\Big)\,. \end{aligned} \tag{3.5}$$

Since we restrict ourselves here to the class of solutions for which for which $c = 0$, and since $f(T) = f(-T)$, one has $r = 0$ and

$$\omega = \operatorname{tr} f'(T) = \operatorname{tr} V'(T) = 0\,. \tag{3.6}$$

Writing $V(T) = \mu f(T)$ and inserting the above ansatz into the equations of motion gives:

$$\lambda\gamma^5 \left(\mu + \frac{3}{4}\mu R^4 + 4d^2R^2\right) + \left(8dqR^2 - 3he^{-d^2/8}R^4\right)\gamma^5 = 0\,, \tag{3.7}$$

for the T equation of motion, and

$$\left(d^2q + \frac{3}{8}\mu q R^2 - \frac{3}{2}hdR^2\right)\left(a\gamma^i + ib\gamma^i\gamma^5\right) = 0\,, \tag{3.8}$$

for the ϕ^i equations of motion, where

$$R^2 = \phi^i\phi^i = 4\left(a^2 + b^2\right) \tag{3.9}$$

is the squared radius of the fuzzy S^3. For solutions of this form note that we can write the above equat0ions of motion as:

$$\begin{aligned}\left(\mu\lambda + \frac{3}{4}\lambda\mu R^4 + 4d^2R^2\lambda + 8dqR^2 - 3he^{-d^2/8}R^4\right)\frac{T}{d} &= 0\,,\\ \left(d^2q + \frac{3}{8}\mu q R^2 - \frac{3}{2}he^{-d^2/8}dR^2\right)X^i &= 0\,.\end{aligned} \tag{3.10}$$

Since $c = 0$,

$$\begin{aligned}\mu q &= e^{-d^2/4}\,,\\ \mu\lambda &= -\frac{d}{2}e^{-d^2/4}\,.\end{aligned} \tag{3.11}$$

Thus if one defines

$$\hat{h} = e^{d^2/8}h\,. \tag{3.12}$$

The equations of motion become,

$$\begin{aligned}-\frac{d}{2} - \frac{3}{8}dR^4 - 2\frac{d^3}{\mu}R^2 + 8\frac{d}{\mu}R^2 - 3\hat{h}R^4 &= 0\,,\\ \frac{d^2}{\mu} + \frac{3}{8}R^2 - \frac{3}{2}\hat{h}dR^2 &= 0\,.\end{aligned} \tag{3.13}$$

These two equations can be solved for $d(h)$ and $R(h)$. The physical solutions require real positive R^2.

The energy of the solutions is given by

$$E = 4q\left(\mu + \frac{3}{4}\mu R^4 + 4d^2R^2\right) - 48hdR^4\,. \tag{3.14}$$

The solutions of interest here are unstable. Therefore unlike the polarization of D0 into D2 discussed in [4] solutions of interest may have higher energy than the trivial solution with $\phi^i = 0$. Both the energies and couplings to Ramond-Ramond forms of these solutions are subject to modifications by higher order terms in the action, which may have to be taken into account in attempts to compare the large-N solutions with D3-branes. We will discuss solutions of the large-N generalization of these equations in section 5.

4. Fuzzy S^3 for general N

One can generalize the above solutions to larger matrices in a manner which gives a commutative S^3 in the large-N limit. This can be done using methods similiar to those used in the construction of the fuzzy S^4 [22], with a few crucial differences. In particular, it neccessary to embed the solutions in certain *reducible* representations of Spin(4). In some sense, the fuzzy S^3 which we obtain can be viewed as a subspace of the fuzzy S^4

For the fuzzy S^4, the matrices G_μ satisfying $G_\mu G_\mu = R^2$ are embedded in the irreducible symmetric tensor representations of Spin(5);

$$G_\mu = \Big(\Gamma_\mu \otimes 1 \otimes 1 \cdots \otimes 1 + 1 \otimes \Gamma_\mu \otimes 1 \cdots \otimes 1 \cdots + 1 \otimes \cdots \otimes 1 \otimes \Gamma_\mu\Big)_{\text{sym}} . \tag{4.1}$$

The index μ runs from 1 to 5. It is convenient to rewrite this as

$$G_\mu = \sum_k \rho_k(\Gamma_\mu) P_n , \tag{4.2}$$

where the right-hand side is a set of operators acting on the n-fold tensor product $V \otimes V \cdots V$. The expression $\rho_k(\Gamma_\mu)$ is the action of Γ_μ on the k-th factor of the tensor product.

$$\rho_k(A)|e_{i_1} e_{i_2} \cdots e_{i_k} \cdots e_{i_n}\rangle = A^{j_k}_{i_k} |e_{i_1} e_{i_2} \cdots e_{j_k} \cdots e_{i_n}\rangle . \tag{4.3}$$

The symmetrization operator P_n is given by $P_n = \sum_{\sigma \in S_n} \frac{1}{n!} \sigma$ where σ acts as:

$$\sigma |e_{i_1} e_{i_2} \cdots e_{i_k} \cdots e_{i_n}\rangle = |e_{i_{\sigma(1)}} e_{i_{\sigma(2)}} \cdots e_{i_{\sigma(n)}}\rangle . \tag{4.4}$$

We wish to construct a fuzzy S^3 with manifest Spin(4) covariance. As a first step in constructing a fuzzy S^3, we will consider the matrices G_i defined above, with index i running from 1 to 4. The symmetric tensor representations of Spin(5) decompose under Spin(4) into a sum of reducible representations. The matrices G_i are maps between irreducible representations of Spin(4). A fuzzy S^3 will be defined by matrices $\hat{G}_i$ whose matrix elements are those of G_i in a direct sum of irreducible Spin(4) representations related by G_i. The irreducible representations in this direct sum will be fixed by requiring $\sum \hat{G}_i \hat{G}_i$ to be proportional to the identity.

Since $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, the irreducible representations are labelled by a pair of spins (j_l, j_r). The fundamental spinor representation V of Spin(5) decomposes under Spin(4) into $(1/2, 0) \oplus (0, 1/2) = P^+V \oplus P^-V$, where P^+ and P^- are positive and negative chirality projectors. The symmetric tensor representations of Spin(5), $\mathrm{Sym}^n(V)$, decompose into irreducible representations of Spin(4) as follows;

$$\mathrm{Sym}(V^{\otimes n}) = \oplus_{k=0}^{n} V_{(\frac{n-k}{2}, k/2)} \,. \tag{4.5}$$

This can be proved by observing

$$\mathrm{Sym}(V^{\otimes n}) = P_n V^{\otimes n} = P_n (P_+ + P_-)^{\otimes n} V^{\otimes n} = P_n \sum_k ({P_+}^{\otimes k} {P_-}^{\otimes n-k})_{\mathrm{sym}} V^{\otimes n} \,. \tag{4.6}$$

The dimension of $\mathrm{Sym}^n(V)$ is $(n+1)(n+2)(n+3)/6$. It is easy to check that the above decomposition is consistent with the dimensions of the representations.

$$\sum_{k=0}^{n} (n-k+1)(k+1) = \frac{(n+1)(n+2)(n+3)}{6} \,. \tag{4.7}$$

Since $\Gamma^i P_+ = P_- \Gamma^i$ and $\Gamma^i P_- = P_+ \Gamma^i$, G_i is a map between different irreducible representations of Spin(4) and vanishes when restricted to a particular irreducible representation. The fuzzy S^3 will be defined the matrices

$$\hat{G}_i = P_{\mathcal{R}} G_i P_{\mathcal{R}} \,, \tag{4.8}$$

where $P_{\mathcal{R}}$ is the projector to a reducible representation $\mathcal{R}$. We require that $\sum_{i=1}^4 G_i G_i$ is proportional to the identity within $\mathcal{R}$. One can show that $\sum_{i=1}^4 G_i G_i$ commutes with the Spin(4) generators $\sum_l \rho_l(\sigma_{jk})$ (see appendix A), where $\sigma_{ij} = \Gamma_i \Gamma_j - \Gamma_j \Gamma_i$. By Schur's lemma, $\sum_i G_i G_i$ is proportional to the identity within each irreducible representation[2] However, the proportionality factor may differ between them. We must therefore take $\mathcal{R}$ to be the direct sum of irreducible representations related by G_i and having the same $\sum_i G_i G_i$.

In the representation $(\frac{n+k}{2}, \frac{n-k}{2})$, one finds (see appendix A) that

$$\sum_i \hat{G}_i \hat{G}_i = 4(n + nk - k^2) \,. \tag{4.9}$$

This is symmetric under $k \to n-k$. However there is no other degeneracy. Thus $\sum G_i G_i$ is proportional to the identity in the representation $(k/2, \frac{n-k}{2}) \oplus (\frac{n-k}{2}, k/2)$. For $\hat{G}_i$ to be non trivial in this representation, G_i must be a map between it's irreducible components. This gives $k = n - k \pm 1$ and

$$\mathcal{R} = \mathcal{R}_+ \oplus \mathcal{R}_- = \left(\frac{n+1}{4}, \frac{n-1}{4}\right) \oplus \left(\frac{n-1}{4}, \frac{n+1}{4}\right) . \tag{4.10}$$

[2]For $n = 1$, $G_i G_i$ is equal to 1 for every i, without any need to sum on i. For large n one must sum over i to obtain something which is proportional to the identity. This is fortunate, since otherwise one could not obtain a classical S^3 in the large-n limit.

The dimension of $\mathcal{R}$ is

$$N = \frac{1}{2}(n+1)(n+3)\,. \tag{4.11}$$

The restriction to this particular sum of Spin(4) irreducible representations essentially amounts to considering a subspace of the fuzzy S^4 of [22], which previously included Spin(4) representations with all values of k. The fuzzy S^4 has

$$\sum_{\mu=1}^{5} G_\mu G_\mu = n^2 + 4n\,. \tag{4.12}$$

For the fuzzy S^3 in the representation $\mathcal{R}$, we find (see appendix A) that

$$\sum_{i=1}^{4} \hat{G}_i \hat{G}_i = n^2 + 4n - 1\,. \tag{4.13}$$

Note that in the large-n limit, the leading term is the same for the S^3 and the S^4. Thus the fuzzy S^3 is a "great sphere" on an equator of the fuzzy S^4. One can easily check that $G_5 G_5 = 1$ when restricted to the representation $\mathcal{R}$.

A general form for a fuzzy S^3 which approaches a classical S^3 in the large-n limit is given by matrices

$$\phi_i = \frac{1}{n}\Big(a\hat{G}_i + ib\hat{G}_i(P_{\mathcal{R}_+} - P_{\mathcal{R}_-})\Big) = \frac{1}{n}(a\hat{G}_i + ib\hat{G}_i\hat{G}_5) \tag{4.14}$$

which is the natural generalization of the $n=1$ expression $\phi_i = a\Gamma_i + ib\Gamma_i\Gamma_5$. Since

$$\hat{G}_i\hat{G}_5 = -\hat{G}_5\hat{G}_i\,, \tag{4.15}$$

it follows that:

$$R^2 = \phi_i\phi_i = (a^2+b^2)\frac{(n^2+4n-1)}{n^2}\,. \tag{4.16}$$

The sphere is classical in the large-n limit because ϕ_i has eigenvalues of order 1, whereas

$$[\phi_i, \phi_j] = \frac{(a^2+b^2)}{n^2} P_{\mathcal{R}} \sum_l \rho_l(\sigma_{ij}) P_{\mathcal{R}} \tag{4.17}$$

has largest eigenvalue of order $1/n$.

5. Large-N equations

We now insert the ansatz $\phi_i = \frac{1}{n}(a\hat{G}_i + ib\hat{G}_i\hat{G}_5)$ into the equations of motion. For the tachyon, we make the ansatz

$$T = c + d(P_{\mathcal{R}_+} - P_{\mathcal{R}_-}) = c + d\hat{G}_5\,. \tag{5.1}$$

With this ansatz, the relations (3.4) and (3.5) still apply with $\gamma_5 \to P_{\mathcal{R}_+} - P_{\mathcal{R}_-} = \hat{G}^5$. Note that for $c = b = 0$, the tachyon behaves like the fifth embedding coordinate of a fuzzy S^4, restricted to a particular Spin(4) representation. Considering the class of solutions with $c = 0$, and using properties described in appendix A, one finds the following equations of motion.

The T equation of motion is;

$$\hat{G}^5 \Bigg[\mu\lambda + \mu\lambda \frac{(n+1)(n+5)}{(n(n+4)-1)^2} R^4 + 2d^2\lambda \frac{(n+1)(n+3)}{n(n+4)-1} R^2 + + 4dq \frac{(n+1)(n+3)}{n(n+4)-1} - 4he^{-d^2/8} \frac{(n+5)(n+1)}{(n(n+4)-1)^2} R^4 \Bigg] = 0 \,. \tag{5.2}$$

The ϕ equations of motion are;

$$\left(d^2 q + \frac{3\mu q}{2} \frac{R^2}{n(n+4)-1} - 2hde^{-d^2/8} \frac{R^2(n+2)}{n(n+4)-1} \right) \phi^i = 0 \,. \tag{5.3}$$

6. Solutions

We discuss here some properties of solutions to the above equations. While these solutions certainly have some of the requisite physical properties, we have no convincing identification with integral numbers of spherical D3-branes, as we will elaborate in the following.

At large n the equations take the form:

$$\begin{aligned} \mu\lambda + 2d^2\lambda R^2 + \frac{\mu\lambda}{n^2} R^4 - \frac{4he^{-d^2/8} R^4}{n^2} + 4dqR^2 &= 0 \,, \\ d^2 q + \frac{3\mu q}{2} \frac{R^2}{n^2} - 2he^{-d^2/8} d \frac{R^2}{n} &= 0 \,. \end{aligned} \tag{6.1}$$

We consider solutions with $d = \tilde{d}/n$ and $R^2 = \tilde{R}^2 n$ with $\tilde{d}$ and $\tilde{R}$ finite in the large-n limit. For the potential $V = e^{-d^2/4}$, one has $\lambda = -\tilde{d}/2n$, and $\mu q = 1$ in the large-n limit. The leading terms in the equations of motion at large n give

$$\tilde{d} = \frac{3}{4h} \tag{6.2}$$

and

$$\hat{R}^2 = \frac{\tilde{d}}{\mu h} \,. \tag{6.3}$$

Since μ is positive, this gives positive R^2.

For finite n, the energy is given by

$$E = q \frac{(n+1)(n+3)}{2} \left[\mu + \frac{(n+1)(n+5)}{(n(n+4)-1)^2} \mu R^4 + 2 \frac{(n+1)(n+3)}{n(n+4)-1} d^2 R^2 \right] - - 2dh \frac{(n+1)^2(n+3)(n+5)}{n^4} \,. \tag{6.4}$$

Similar remarks as in the $n = 1$ case regarding stability and corrections apply here.

6.1 Ramond-Ramond couplings

The relation to spherical D3-branes can be explored by a simple argument analogous to the discussion of fuzzy four-spheres [22]. The net brane-charge is zero, however one can determine the number of spherical D3-branes by computing a charge contribution from a single hemisphere.

Recalling [21] that the 3-brane potential couples to the zero-brane action through the term

$$\int dt[\phi_1, \phi_2][\phi_3, T]C_{0123}\,. \tag{6.5}$$

The calculation of the charge of a single hemisphere is closely analogous to that which gives the 4-brane charge in the case of zero-branes polarizing to fuzzy 4-spheres [22]. In our case we have

$$\frac{(2\pi)^4}{8\pi^2}\epsilon_{ijkl}\,\mathrm{Tr}_{1/2}\,\phi^j\phi^k[\phi^l, T]\,, \tag{6.6}$$

where $\mathrm{Tr}_{1/2}$ indicates that the trace is evaluated in the subspace for which the eigenvalues of ϕ^i are positive. At large n, one finds this to be equal to $2d$ times the volume of the $\phi^i > 0$ hemisphere, where d is the quantity which enters our ansatz for the tachyon (5.1).

$$\begin{aligned} 2\pi^2\epsilon_{ijkl}\,\mathrm{Tr}_{1/2}\,\phi^j\phi^k[\phi^l, T] &= 4\pi^2 d\frac{(n+2)}{n^3}\,\mathrm{Tr}_{1/2}\,G^iR^3 \\ \longrightarrow d4\pi^2R^3 &= 2dVol_{1/2}\,. \end{aligned} \tag{6.7}$$

Thus it appears that d is proportional to the number of D3-branes.

Alternatively, one may compute the dipole coupling of of a spherical D3-brane to the Ramond-Ramond four-form and compare it to the coupling of the D0 solution. The couping of Q_3 spherical D3-branes to the a constant RR 5-form field strength of the form (2.1) is given by

$$\begin{aligned} \mu_3 Q_3\int dt\int_{S^3} PB\left(C^{(4)}\right) &= \mu_3 Q_3\int dt\int_{S^3} h\epsilon_{ijkl}X^i\frac{\partial X^j}{\partial\sigma^\alpha}\frac{\partial X^k}{\partial\sigma^\beta}\frac{\partial X^l}{\partial\sigma^\gamma}d\sigma^\alpha\wedge d\sigma^\beta\wedge d\sigma^\gamma \\ &= 2\pi^2\mu_3 Q_3 hR^4\,. \end{aligned} \tag{6.8}$$

On the other hand the coupling arising for the D0 configuration is proportional to

$$h\mu_0\int dt\epsilon_{ijkl}\,\mathrm{Tr}\left[[\phi^i, T]\phi^j\phi^k\phi^l\right] \tag{6.9}$$

which for our ansatz (4.14) and (5.1) becomes

$$\int dt\epsilon_{ijkl}\,\mathrm{Tr}\left[2d\phi^i\phi^j\phi^k\phi^l\right] = 2dhR^4\frac{(n+1)^2(n+3)(n+5)}{(n^2+4n-1)^2} \tag{6.10}$$

or $2dhR^4$ for $n\to\infty$.

Therefore (up to a numerical factor) the quantity d which appears in the Tachyon ansatz (5.1), would be identified with the number of spherical three-branes if such a dual description existed. However, for the large-n solutions which we have found, this quantity is zero as $n \to \infty$. Thus we cannot identify these large-n solutions with spherical D3-branes. Note that for the fuzzy S^3 ansatz, there are continuous classes of solutions, with d depending on h. In this case, one would never expect a quantization of the dipole moment consistent with a description in terms of integer numbers of D3-branes. Perhaps this situation changes if one considers probe D0-branes in a consistent IIB supergravity background, such as $ADS_5 \times S_5$ with quantized five-form flux on the S^5. The polarization of stable branes in an a consistent AdS background was considered in [23].

7. Other odd non-commutative spheres

The basic ingredients that went into our construction of the non-commutative three-sphere admit a number of generalizations. To understand this it is useful to reformulate our discussion of the non-commutative 3-sphere in a more abstract form. Essentially we needed objects X^{μ} as matrices in some family of representations of SO(4) parametrized by an integer n. They were identified with maps between representations R_1 and R_2. R_1 was an irreducible representation of SO(4) of positive chirality, i.e. a representation where Γ_5 takes eigenvalue 1. R_2 was a representation of negative chirality, i.e. a representation where Γ_5 takes eigenvalue -1. We needed a non-zero coupling between R_1, R_2 with the vector representation of Spin(4). We may think of X as the Clebsch-Gordan coupling $R_1 \otimes R_2$ to the vector representation.

Two natural generalizations are suggested by the above. One is to consider odd spheres of other dimensions. For example for 5-spheres we would like to take two representations of Spin(6) satisfying the above conditions. Since Spin(6) is isomorphic to SU(4), we can exploit this to give a simple description of the desired representations. The two chiral four-dimensional representations of Spin(6) can be identified with the fundamental and the anti-fundamental of SU(4). These are represented in Young Tableau notation by a single box, and a column of three boxes respectively. We can take the representation R_1 to be given by the Young Tableau with n columns of length 3, and $n-1$ columns of length 1. The representation R_2 can be taken to be given by the Young Tableau with $n-1$ columns of length 3 and n columns of length 1.

Another generalization would to be consider q-deformation. We could take as our definition of X the Clebsch-Gordans of the appropriate quantum group. It would be interesting to compare such a construction with the quantum spheres of [24] for example.

8. Summary and outlook

We have found matrices defining non-commutative 3-spheres with a Spin(4) invariance. This construction is applicable for matrices of size $N = (n+1)(n+3)/2$, where n is odd. These three-spheres can be viewed as a subspace of the non-commutative four-sphere defined in [22]. We have also proposed a general construction of higher-dimensional odd fuzzy spheres along these lines. While the construction of [22] gave a description of non-commutative even spheres as solutions to Matrix Theory (see also [25] for a variety of fuzzy spheres appearing in Matrix Theory), the description of fluctations of fuzzy spherical branes is left as an open problem. This should be related to the problem of giving a complete characterization of the algebra of functions and differential calculus on the non-commutative sphere. Likewise, we leave for future research the description, in terms of non-commutative geometry, of the fluctuations of the solutions we described. The techniques of [26] may be useful here.

We have obtained unstable solutions of the IIB D0-brane equations of motion in the presence of a background RR-5 form background. The adjoint scalars are given by the matrices defining the fuzzy S^3, and the tachyon is related to the fifth matrix coordinate involved in the definition of the fuzzy S^4 in which the S^3 is embedded. The tachyon as an extra dimension has been hinted at in several papers [8] and the full significance of this intriguing geometrical behaviour remains to be understood.

At finite n, the solutions we find carry an apparent D3 dipole moment. In the large-n limit, the non-commutative S^3 becomes a classical S^3 and the dipole moment vanishes. Thus, contrary to what one may have expected, we can not view these solutions as spherical D3-branes. However, this may simply be a consequence of having considered an inconsistent supergravity background. Note that if one did find solutions which could be regarded as D3-branes on a classical S^3, then it seems likely that the D0-branes could be described by sphalerons in a dual D3 description. The correspondance between sphalerons and unstable branes has been discussed, in a different context in [27, 28].

It would be very interesting to understand the mechanism which might lead to D3-charge quantized in the standard way. The choice of a consistent supergravity background mentioned above, the geometrical nature of the tachyon in relation to fuzzy 4-spheres, as well as detailed properties of the tachyon dependent couplings to Ramond-Ramond forms in unstable branes might be expected to enter the story.

Odd spheres also appear in the ADS/CFT context and non-commutative versions based on quantum groups have been studied in the context of the stringy exclusion principle [29, 30, 31]. The investigation of the relation between quantum 3-sphere constructions and the non-commutative 3-sphere in this paper is left for the future. Techniques like those of [32] where a connection between fuzzy 2-sphere and q-2sphere was established may be useful. A different context where odd non-commutative spheres have been discussed recently is [33]. A relation between

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non-commutative S^4 and quantum S^3 also appears in [26]. Earlier discussions of fuzzy spheres appear in [34]. It will be undoubtedly illuminating to undesrtand the physical and mathematical relations between these different constructions of non-comuutative spheres.

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A. Useful formulae

We describe here some useful formulae. In this section Roman indicies are taken to run from 1 to 4 and Greek indicies to run from 1 to 5. Consider the symmetric tensor product representations of Spin(5):

$$\mathrm{Sym}(V^{\otimes n}) = \Big(V_1 \otimes V_2 \otimes V_3 \cdots \otimes V_n\Big)_{\mathrm{sym}}, \tag{A.1}$$

where each V_α is a 4-dimensional spinor and the symmetrization is as defined in (4.4). The generators of Spin(5) are

$$G_{\mu\nu} = \sum_{l=1}^{n} \rho_l(\sigma_{\mu\nu}) = \sigma_{\mu\nu} \otimes 1 \otimes 1 \cdots + 1 \otimes \sigma_{\mu\nu} \otimes 1 \cdots . \tag{A.2}$$

It is easy to verify that

$$\left[\sum_{\mu=1}^{5} \Gamma^\mu \otimes \Gamma^\mu \otimes 1 \otimes 1 \cdots, G_{\mu\nu}\right] = 0 . \tag{A.3}$$

Thus in any irreducible representation of Spin(5) the matrix $\sum_{\mu=1}^{5} \Gamma^\mu \otimes \Gamma^\mu \otimes 1 \otimes 1 \cdots$ is proportional to the identity. In the symmetric tensor representations, this proportionality factor is one, as can be verified by multiplying the whole expression on the left by $\sum_{\mu=1}^{5} \Gamma^\mu \otimes \Gamma^\mu \otimes 1 \otimes 1 \cdots$ twice and deriving a cubic equation for the constant using Γ-matrix identities. Using this, one finds that matrices G^μ defined in the symmetric tensor representations of Spin(5) by

$$G^\mu = \sum_{l=1}^{n} \rho_l(\Gamma^\mu) = \Big(\Gamma^\mu \otimes 1 \otimes 1 \cdots + 1 \otimes \Gamma^\mu \otimes 1 \cdots + \cdots\Big)_{\mathrm{sym}} \tag{A.4}$$

satisfy the property

$$\sum_{\mu=1}^{5} G^\mu G^\mu = n^2 + 4n . \tag{A.5}$$

The symmetric tensor representations of Spin(5) decompose under Spin(4) as in (4.5) and (4.6);

$$\mathrm{Sym}(V^{\otimes n}) = P_n \sum_k (P_+{}^{\otimes k} P_-{}^{\otimes n-k})_{\mathrm{sym}} V^{\otimes n} \,. \tag{A.6}$$

The matrices of the form

$$\sum_{i=1}^{4} \Gamma^i \otimes \Gamma^i \otimes 1 \otimes 1 \cdots = 1 - \Gamma^5 \otimes \Gamma^5 \otimes 1 \otimes 1 \cdots \tag{A.7}$$

commute with all the Spin(4) generators G_{ij}. If one restricts to one of the Spin(4) irreducible representations labelled by k, then this quantity is proportional to the identity by Schur's Lemma. Using (A.7) it can be shown that

$$\sum_{i=1}^{4} G^i G^i = 4\left(n^2 + nk - k^2\right), \tag{A.8}$$

when restricted to the representation labelled by k.

Let us now consider the reducible Spin(4) representation $(\frac{n+1}{4}, \frac{n-1}{4}) \oplus (\frac{n-1}{4}, \frac{n-1}{4})$ which is the sum of representations with $k = (n+1)/2$ and $k = (n-1)/2$ with odd n. We write the restriction of G^μ to this reducible representation as $\hat{G}^\mu$. Taking $T = c + d\hat{G}^5$ and $f(T) = q + r\hat{G}^5$, one obtains the following relations for the terms which appear in the equations of motion.

$$\begin{aligned}
\hat{G}^i \hat{G}^i &= n^2 + 4n - 1\,,\\
[\hat{G}^i, \hat{G}^j][\hat{G}^i, \hat{G}^j] &= -4(n+1)(n+5)\,,\\
[\hat{G}^i, T][\hat{G}^i, T] &= -2d^2(n+1)(n+3)\,,\\
\Big[\hat{G}^i, \{[\hat{G}^i, T], f(T)\}\Big] &= 4dq(n+1)(n+3)\hat{G}_5\,,\\
\epsilon_{ijkl}\hat{G}^i\hat{G}^j\hat{G}^k\hat{G}^l &= 2(n+1)(n+5)\hat{G}_5\,,\\
\Big[\{f(T), [\hat{G}^i, T]\}, T\Big] &= 8d^2 q \hat{G}^i\,,\\
\Big[\{f(T), [\hat{G}^i, \hat{G}^k]\}, \hat{G}^k\Big] &= 24q\hat{G}^i\,,\\
\epsilon_{ijkl}\{[\hat{G}^j, \hat{G}^k], [\hat{G}^l, T]\} &= 16d(n+2)\hat{G}^i\,.
\end{aligned} \tag{A.9}$$

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